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S.I. : ANALYTICAL MODELS FOR FINANCIAL MODELING AND RISK MANAGEMENT

Robust equity portfolio performance

Jang Ho Kim[1] · Woo Chang Kim[2] · Do-Gyun Kwon[2] · Frank J. Fabozzi[3]

Abstract The earliest documented analytical approach to portfolio selection is Markowitz's mean–variance analysis, which attempts to find the portfolio with optimal performance by considering the tradeoff between return and risk. The performance of mean–variance analysis has been the subject of many studies and compared to other portfolio construction approaches such as a naïve equally-weighted allocation scheme. In recent years, several approaches have been proposed to improve the mean–variance model by reducing the sensitivity of the portfolio selection process in order achieve robust performance. Although robust portfolio optimization has been one of the most researched methods for improving portfolio robustness, the performance of robust portfolios has not been the major focus of studies. In this paper, a comprehensive analysis on robust portfolio performance is presented for equity portfolios constructed in the U.S. market during the period 1980 and 2014, and results confirm the advantage of robust portfolio optimization for controlling uncertainty while efficiently allocating investments.

Keywords Portfolio optimization · Robust optimization · Portfolio performance · U.S. equity market

✉ Jang Ho Kim
janghokim@khu.ac.kr

Woo Chang Kim
wkim@kaist.ac.kr

Do-Gyun Kwon
dogyun@kaist.ac.kr

Frank J. Fabozzi
fabozzi321@aol.com

1 Kyung Hee University, Yongin-si, South Korea

2 Korea Advanced Institute of Science and Technology (KAIST), Daejeon, South Korea

3 EDHEC Business School, Nice, France

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1 Introduction

Portfolio performance is certainly the most important part of portfolio management. Regardless of how a portfolio is constructed or which asset classes a portfolio is invested in, it is likely that a portfolio manager is not criticized if the risk-adjusted performance net of fees exceeds the benchmark return. One of the earliest frameworks for portfolio management is mean–variance analysis (Markowitz 1952). This approach to portfolio construction considers expected returns and variance/covariances of returns for optimizing the portfolio that is likely to have the best performance, described as the portfolio with high return and low risk, and the model sparked much research in portfolio selection (see, for example, Fabozzi et al. 2002; Kolm et al. 2014). Along with the theoretical developments, performances of standard mean–variance portfolios (Cohen and Pogue 1967; Bloomfield et al. 1977; Jorion 1992), constrained portfolios (Frost and Savarino 1988; Grauer and Shen 2000), the minimum-variance portfolio (Haugen and Baker 1991; Clarke et al. 2006), the equally-weighted portfolio (DeMiguel et al. 2009), and mean–variance portfolio using factor models (Fan et al. 2008) have also been analyzed.

Observations on portfolio performance revealed a major drawback of the classical mean–variance model: the sensitivity of portfolio weights and returns to the model inputs (Michaud 1989; Best and Grauer 1991; Broadie 1993). While risk is considered in mean–variance optimization, the distinction between risk and uncertainty leads to new approaches for managing uncertain situations (see, for example, Hansen and Sargent 2008; Sargent 2014). One popular approach for resolving this shortcoming is robust portfolio optimization, which applies the worst-case approach of robust optimization to portfolio selection (Fabozzi et al. 2010; Kim et al. 2014). While many robust formulations have been developed and the robustness attribute of robust portfolios is generally understood, to the best of our knowledge, there have not been notable attempts to examine the overall performance of portfolios constructed from robust portfolio optimization. Scherer (2007) tests out-of-sample performance of robust portfolios from simulation but only compares the expected utility of portfolios. Others have discussed the computational results of robust portfolio optimization performance, but most experiments are included as numerical examples (see, for example, Goldfarb and Iyengar 2003; Tütüncü and Koenig 2004; Ceria and Stubbs 2006).

Although the theoretical developments of robust portfolio optimization explain its advancement in reducing the sensitivity of portfolios, a thorough investigation of actual performance should be carried out to validate its practical use by investment managers. Therefore, in this paper, we present a comprehensive analysis of the performance of portfolios formed using robust portfolio optimization. While there are many advanced robust portfolio optimization models, we focus on a number of basic robust formulations based on the classical mean–variance model. The main contribution of the paper is to examine if even the simplest robust portfolio models achieve robust performance compared to other portfolio strategies. The empirical results presented here provide evidence that robust portfolio optimization forms portfolios that are superior at reducing worst-case loss and efficiently allocating risk.

The historical performance of robust portfolios in the U.S. equity market from 1980 to 2014 is observed. Although some performance details may be dependent on the selected data and time period, the long-term historical performance exhibited through an extended list of performance measures in this study will provide the grounds for discussing robust portfolio performance. Finally, we note that the recent growth in automated investment management leads to the inevitability of utilizing portfolio optimization models and the efficiency of applying robust portfolio optimization also becomes a critical discussion especially since robustness is crucial for managing long-term investments.

The remainder of the paper is organized as follows. Section 2 describes the basics of robust portfolio optimization. Section 3 explains the details of the portfolio evaluation methodology including portfolio construction, rebalancing, performance measures, and data description. Overall performance results from 1980 to 2014 are presented in Sect. 4, while Sect. 5 discusses sub-period performances as well as analyses on yearly returns. Section 6 concludes the paper.

2 Robust portfolio optimization

The main advantage of robust portfolio optimization is that portfolios with increased robustness are formed by solving optimization problems that are based on the classical mean–variance problem. Thus, applying robust optimization becomes a natural extension for achieving stable performance for mean–variance investors. More importantly, many robust counterparts of the classical portfolio problem are formulated as optimization problems that are solved efficiently (see, for example, Fabozzi et al. 2007a, b; Kim et al. 2016).

The classical mean–variance model proposed by Markowitz (1952) finds the optimal portfolio using the mean and variance of portfolio returns. Among several approaches, the formulation that finds the portfolio with maximum expected return with a given level of variance is,

$$\begin{aligned} \max_{\omega} \quad & \mu^T \omega \\ \text{s.t.} \quad & \omega^T \Sigma \omega = \sigma_p^2 \\ & \omega^T \iota = 1 \end{aligned} \tag{1}$$

where $\omega \in \mathbb{R}^n$ is the portfolio weight of n assets, $\mu \in \mathbb{R}^n$ is the expected return of the assets, $\Sigma \in \mathbb{R}^{n \times n}$ is the covariance matrix of asset returns, $\sigma_p^2 \in \mathbb{R}$ is the desired level of portfolio variance, and $\iota \in \mathbb{R}^n$ is the vector of ones. Another formulation that uses the tradeoff between risk and return as the objective function is

$$\begin{aligned} \min_{\omega} \quad & \omega^T \Sigma \omega - \lambda \mu^T \omega \\ \text{s.t.} \quad & \omega^T \iota = 1 \end{aligned} \tag{2}$$

where $\lambda \in \mathbb{R}$ is the coefficient that represents the risk appetite of an investor and setting it to zero finds the global minimum-variance (GMV) portfolio. As opposed to the formulation given by (1), formulation (2) is a minimization problem because the objective subtracts expected return from the portfolio's variance.

While the above methods are intuitive, as explained earlier, one concern is the uncertainty of the input parameters. The true distribution of asset returns is unknown and the values of the means, variances, and covariances of asset returns can only be estimated when making investment decisions and the realized values may be significantly different from their estimates.

In robust portfolio optimization, uncertain parameters are assumed to be within a set of possible values. The optimal robust portfolio is the best choice when all values within the uncertainty set are considered. The robust counterpart of the formulation given by (2) is written as

$$\min_{\omega} \max_{(\mu, \Sigma) \in \mathcal{U}} \omega^T \Sigma \omega - \lambda \mu^T \omega$$

$$\text{s.t. } \omega^T \iota = 1 \tag{3}$$

where $\mathcal{U}$ is the uncertainty set for the two inputs, μ and Σ. Since errors in expected returns of assets are known to affect portfolios much more than errors in variances or covariances (Chopra and Ziemba 1993), robust portfolios are often computed by only incorporating uncertainty in mean returns from an uncertainty set of possible mean vector values (Kim et al. 2017). The analysis in this paper also assumes that uncertainty is only contained in mean returns.

Common approaches for defining uncertainty sets for expected returns include setting intervals for the expected return for each asset and setting a combined ellipsoidal set. The first approach using intervals, also known as box or interval uncertainty sets, is expressed as

$$\mathcal{U}_\delta\left(\hat{\mu}\right) = \left\{\mu \middle| \left|\mu_i - \hat{\mu}_i\right| \le \delta_i, \quad i = 1, \ldots, n\right\} \tag{4}$$

where $\hat{\mu} \in \mathbb{R}^n$ is an estimate of the mean vector and $\delta \in \mathbb{R}^n$ sets the possible deviation from the estimated value for each asset. Therefore, the box uncertainty set given by (4) specifies an interval around an estimator for the expected asset return. The second uncertainty set forms an ellipsoid around an estimated expected return vector,

$$\mathcal{U}_\kappa\left(\hat{\mu}\right) = \left\{\mu \middle| \left(\mu - \hat{\mu}\right)^T \Sigma_\mu^{-1} \left(\mu - \hat{\mu}\right) \le \kappa^2\right\} \tag{5}$$

where $\kappa \in \mathbb{R}$ controls the size of the ellipsoid and $\Sigma_\mu \in \mathbb{R}^{n \times n}$ is the covariance matrix of estimation errors.

The robust problem given by (3) can be reformulated as a tractable optimization problem when defining the uncertainty set to be either a box uncertainty set given by (4) or an ellipsoidal uncertainty set given by (5). The robust counterpart of (3) with the uncertainty set defined as (4) is formulated as[1]

$$\begin{aligned} \min_{\omega} \quad & \omega^T \Sigma \omega - \lambda \left(\hat{\mu}^T \omega - \delta^T \left|\omega\right|\right) \\ \text{s.t.} \quad & \omega^T \iota = 1, \end{aligned} \tag{6}$$

and the robust formulation with the uncertainty set defined as (5) is written as

$$\begin{aligned} \min_{\omega} \quad & \omega^T \Sigma \omega - \lambda \left(\hat{\mu}^T \omega - \kappa \sqrt{\omega^T \Sigma_\mu \omega}\right) \\ \text{s.t.} \quad & \omega^T \iota = 1. \end{aligned} \tag{7}$$

One approach for setting the values of δ and κ is to consider confidence intervals around the estimated $\hat{\mu}$. In our analyses, historical returns during the estimated period are used for forming uncertainty sets with a 95% confidence level with normality assumptions on asset returns. Furthermore, for simplicity in the remaining sections, robust portfolios constructed from formulations (6) and (7) are abbreviated as RB and RE, respectively.

3 Performance evaluation methodology

The goal of this paper is to present a comprehensive analysis of the portfolio constructed using robust portfolio optimization. The details about the performance evaluation settings and methodology are described in this section.

[1] Derivations of formulations (6) and (7) are presented in Fabozzi et al. (2007b) and Kim et al. (2016). These robust formulations can be solved using optimization software.

3.1 Investment period

The two important criteria for selecting the investment period for this experiment are the existence of market crashes during the period and the length of the overall investment horizon. Although investing in robust portfolios is more advantageous during periods of high volatility or market downturn, the investment period for the analysis should be long enough to not only include market crashes but also various market conditions. Hence, this study investigates portfolio performance between 1980 and 2014, a period that includes notable volatile periods such as the collapse of the dot-com bubble in the early 2000s and the global financial crisis that began in 2008. Furthermore, the 35-year period is long enough to evaluate the long-term performance of robust portfolios.

3.2 Data

The experiment is performed on the U.S. equity market; a developed market is selected in order to reduce the influence of trading aspects such as liquidity and currency that are not directly handled by the classical mean–variance model but can affect the optimal investment decision.

The 49 industry portfolios of the U.S. market are used as candidate assets because they present a complete representation of the U.S. stock market. The industry portfolios, which are provided by the data library of Kenneth R. French, are constructed by assigning each stock traded on NYSE, AMEX, and NASDAQ to an industry portfolio every year based on the four-digit SIC code at that time (retrieved from either Compustat or CRSP).[2] The industry portfolios are reconstructed once a year, and the version that computes value-weighted returns and includes dividends are chosen for this experiment. Moreover, the 49 industry portfolios are used as candidate assets because they divide the stock market into a large number of industries that will provide potential for diversification; the 49 industries allow portfolios to capture diversification benefits and the results will also provide insight on robust portfolio performance when investing in individual stocks.

For the returns of a composite index of the U.S. market, the returns are derived from excess market return data provided by French's data library. The excess market return is a value-weighted return of all stocks traded on the NYSE, AMEX, and NASDAQ that are available from CRSP. The risk-free rates are also collected from the same data library, which is the 1-month Treasury bill rate from Ibbotson Associates.

3.3 List of portfolios

Since robust portfolios are developed to resolve the sensitivity issue of mean–variance portfolios, the primary comparison is between portfolios formed from the classical mean–variance optimization and robust portfolio optimization. Mean–variance portfolios with annualized volatility (i.e., annualized standard deviation of returns) of 15, 20, and 25% are observed, constructed using the formulation given by (1).[3] Three corresponding robust portfolios with the same levels of risk appetite as the three classical mean–variance portfolios are also constructed at each rebalancing period. Three portfolios each from the mean–variance model and the robust approach are observed to compile performance at various levels.

[2] The industry returns are available at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

[3] Portfolios with smaller annualized volatility are not considered since the GMV portfolio often shows annualized standard deviation above 10% for estimation periods of 1 year or longer.

Table 1 List of portfolios and abbreviations that are evaluated

(1) Index	Passive strategy that invests in the composite index of the market
(2) EqW	Equally-weighted portfolio of all candidate assets
(3) GMV	Global minimum-variance portfolio
(4) MV15	Classical mean–variance optimal portfolio with annualized standard deviation of 15%
(5) MV20	Classical mean–variance optimal portfolio with annualized standard deviation of 20%
(6) MV25	Classical mean–variance optimal portfolio with annualized standard deviation of 25%
(7) RB15 (8) RB20 (9) RB25	Robust portfolios with box uncertainty set and the same level of risk-averse coefficient as MV15, MV20, and MV25, respectively
(10) RE15 (11) RE20 (12) RE25	Robust portfolios with ellipsoidal uncertainty set where the estimation error covariance matrix is estimated as $\frac{1}{T}\Sigma$ and with the same level of risk-averse coefficient as MV15, MV20, and MV25, respectively (T is the number of return observations and Σ is the covariance matrix of returns)
(13) REf15 (14) REf20 (15) REf25	Robust portfolios with ellipsoidal uncertainty set where the estimation error covariance matrix is estimated from the factor model and with the same level of risk-averse coefficient as MV15, MV20, and MV25, respectively
(16) RE15d (17) RE20d (18) RE25d	Robust portfolios same as RE15, RE20, and RE25, respectively, but with the estimation error covariance matrix assumed as a diagonal matrix

The performance of robust portfolios is also compared against various conventional approaches, typically evaluated relative to the returns of a designated benchmark because it reflects what can be earned from a passive portfolio strategy. The benchmarks used here are a composite equity market index and an equally-weighted portfolio because it obtains diversification without requiring any optimization. Furthermore, the GMV portfolio provides a valuable comparison because it is the portfolio in the mean–variance framework with the lowest risk. In the following sections, investing in a benchmark that is an index, investing in an equally-weighted portfolio, and investing in the GMV portfolio are referred to as the three conservative benchmarks.

More importantly, the performance of many robust portfolios is investigated in order to confirm that observations are not dependent on the specifications of the robust portfolio construction. Performance is collected for robust portfolios with several levels of risk coefficient and different ways for constructing uncertainty sets. For example, there are a number of approaches for calculating the estimation error covariance matrix Σ_μ for the ellipsoidal uncertainty set; the covariance matrix of estimation errors can be estimated by dividing historical returns into estimation and evaluation periods or by observing the residuals of a factor analysis.[4] A simplified formula that estimates the error covariance matrix as a diagonal matrix is also investigated. The full list of portfolios observed along with their abbreviations and details are summarized in Table 1.

[4] The calculation involved in estimating the estimation error covariance matrix is derived in Stubbs and Vance (2005).

3.4 Portfolio rebalancing

In this analysis, portfolios are rebalanced every month. In other words, new optimal portfolio weights are calculated at the beginning of each month and updated accordingly. Investors considering robust portfolios will not rebalance frequently since robust portfolios are less sensitive to changes in the market and the aim is not to aggressively chase growth-potential assets. When rebalancing a portfolio, another key component is the estimation period, which refers to how much historical data are used for estimating parameters when re-optimizing portfolios. In this experiment, estimation periods of 12 and 24 months are selected. Since estimation periods have a direct impact on optimal portfolio weights, results are collected for two different estimation periods. Daily returns during the estimation period are used for forming optimal portfolios at each rebalancing period in order to have enough data points for parameter estimation.

In short, the analysis relies on a rolling-sample approach and this simulates a real investment situation. A portfolio is rebalanced every month and performance of the portfolio is evaluated once monthly returns are collected for the entire investment period.

3.5 Performance measures

The performance of robust portfolios is investigated using a variety of measures that are regularly utilized by portfolio managers when reporting performance. The performance measures that are discussed in Sects. 4 and 5 are introduced below.[5]

- *Holding period return* Total return of a portfolio during its investment period. In this analysis, the initial and final portfolio values are used for representing holding period returns.
- *Annual return* Return of a portfolio expressed as an annualized value. Returns are compounded when calculating the annual return of a portfolio in this analysis.
- *Volatility* Standard deviation of portfolio returns. In this experiment, the standard deviation of monthly returns is computed without annualizing.
- *Alpha* (*Jensen's alpha*) Additional return of a portfolio compared to its theoretical return estimated from capital asset pricing model (Jensen 1968). While alpha is considered a risk-adjusted value, alpha estimated from monthly returns is presented along with annual returns in Sect. 4.
- *Sharpe ratio* Ratio of excess return per unit of risk (Sharpe 1966). Here, the excess annual return, which accounts for compounding, is divided by the annualized volatility of a portfolio.
- *Sortino ratio* Modification of the Sharpe ratio by measuring performance relative to a minimum acceptable return (MAR), which is the target return (Sortino and Price 1994). Average excess return is measured relative to MAR and risk is calculated as the downside risk relative to MAR. In this analysis, Sortino ratio with 0% MAR is assessed.
- *Maximum drawdown* Maximum decline, expressed as a percent change, in portfolio value from a peak during the investment period (Garcia and Gould 1987).
- *Value at risk* (*VaR*) Minimum level of loss that is expected to occur with a certain probability (Linsmeier and Pearson 2000). This experiment estimates the 1-month VaR and the value is expressed as a loss (thus, most values are positive representing negative returns).

[5] An overview on evaluating portfolio performance is included in Chapter 12 by Maginn et al. (2007), and implementing various performance measures are detailed in Kim et al. (2016).

- *Conditional value at risk* (*CVaR*) Extension of VaR where the expected loss beyond the VaR level is found (Rockafellar and Uryasev 2000). The 1-month CVaR is also expressed as a loss in this analysis.
- *Turnover ratio* Proportion of portfolio traded. Among several approaches for measuring turnover, the sum of absolute change in weight for each security is collected at every rebalancing date and the average monthly value is compared (Qian et al. 2007). (In other words, both buying and selling are counted, which is sometimes referred as two-way turnover.)
- *Tracking error* Standard deviation of the differences between a portfolio and its benchmark returns (Roll 1992). Tracking error is calculated using monthly returns in our experiment.

The above 11 measures reflect all aspects of portfolio performance including absolute, relative, risk-adjusted, and worst-case performances.

4 Overall performance from 1980 to 2014

The results of robust portfolio performance from 1980 to 2014 are presented in this section. The findings are summarized into five categories: return, risk, risk-adjusted return, worst-case loss, and additional results. The overall performance during the entire investment period is presented first, followed by more detailed analyses on annual returns and sub-period returns in the next section.

Before presenting the results, it is important to note that forming classical mean–variance portfolio with annualized standard deviation of 15% (denoted as MV15) is not always possible if the estimation period contains volatile returns, especially during 2008 and 2009. When this is the case, the MV15 portfolio is set to have the same weight as the GMV portfolio until the next rebalancing date. The same applies to the corresponding robust portfolios with the same level of risk coefficient as MV15. This occurs less than 2% of the time but must be accounted for when analyzing the performance of the MV15 portfolio; while the weights of the GMV and robust portfolios show similar compositions, mean–variance portfolio weights deviate further from the weights of the GMV portfolio (Kim et al. 2013).

4.1 Returns

The returns of portfolios are computed without considering transaction costs, taxes, and inflation. Thus, even though the values may be higher than the real return of the portfolios (see, for example, Siegel 1992), the results are sufficient for comparing and ranking the performance of various approaches.

The analysis begins by plotting how the value of the portfolios changes from 1980 to 2014, which reflects the holding period return. Figure 1 plots the wealth (logarithm values) generated by the different portfolios when a value of one is invested in portfolios at the beginning of 1980. As shown in both panels of Fig. 1 representing different estimation periods, investment in mean–variance portfolios will end with the highest values in 2014 and thus having the highest holding period returns. The higher return of mean–variance portfolios, which are the more aggressive strategies, is expected because the U.S. stock market had an overall upward trend during the investigation period as evidenced from the performance of the market index. The more interesting finding is that robust portfolios cumulate higher

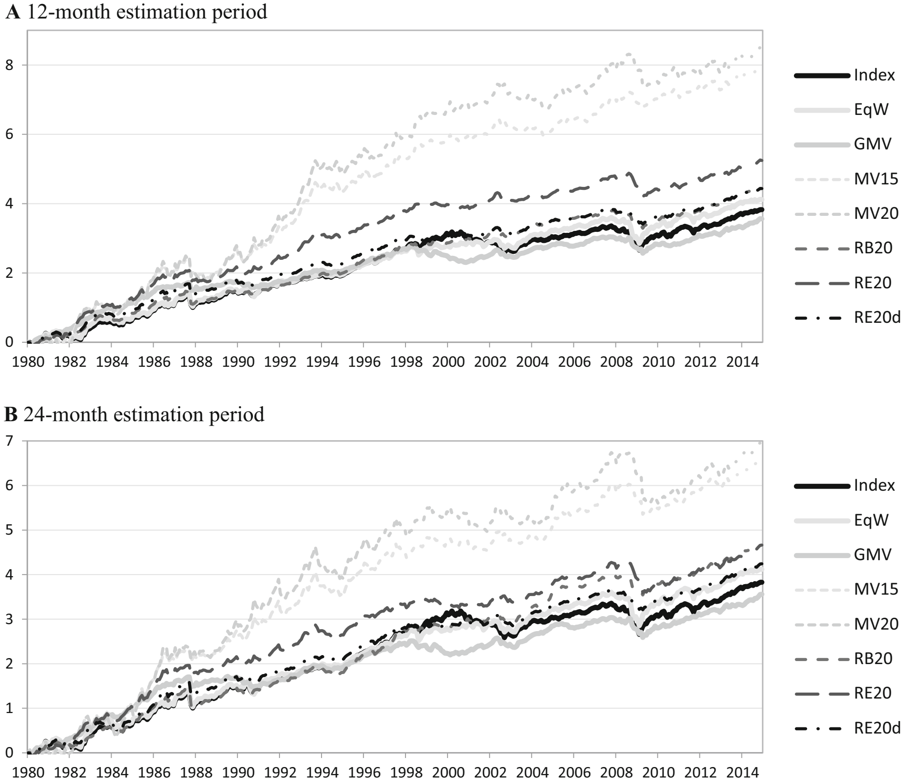

Fig. 1 Total value of portfolios from 1980 to 2014 (logarithm of wealth)

values than the conservative benchmark portfolios. While Fig. 1 only plots a few portfolios, all robust portfolios see higher growth than the conservative benchmarks.

The high long-term return of robust portfolios is confirmed when measuring annual return and alpha. As shown in the first row of Tables 2 and 3, while mean–variance portfolios have the highest annual return, robust portfolios are more attractive than the other three benchmarks regardless of the estimation period selected. A similar pattern is exhibited for the alpha computed from monthly returns. The alpha of investing in the index is zero because the return of that index is considered as the return of the overall market when calculating the value of alpha. Regression analysis, which was used for computing alpha, showed statistical significance at the 1% level in all cases except for MV25 with a 12-month estimation period, which was significant at the 5% level. These show that higher returns can be expected from robust portfolios in most cases than more common conservative approaches for long-term investments.

4.2 Risk

Tables 2 and 3 also include the volatility of portfolios calculated as the standard deviation of monthly returns. It is clear that mean–variance portfolios contain higher risk due to the allocation to more volatile assets in order to attain higher expected return. While GMV portfolios have the lowest volatility, there is not enough evidence to conclude that the index

Table 2 Overall performance of portfolios with 12-month estimation periods during 1980–2014

	Index	EqW	GMV	MV15	MV20	MV25	RB15	RB20	RB25	RE15	RE20	RE25	REf15	REf20	REf25	RE15d	RE20d	RE25d
Annual return	11.8%	12.7%	10.6%	25.7%	27.9%	29.3%	13.2%	13.8%	14.1%	16.2%	16.2%	16.6%	14.5%	14.4%	14.7%	13.5%	13.7%	13.9%
Alpha	–	0.08%	0.28%	1.60%	2.03%	2.48%	0.36%	0.39%	0.40%	0.74%	0.77%	0.82%	0.49%	0.48%	0.50%	0.33%	0.32%	0.31%
Volatility	4.5%	4.7%	3.3%	7.6%	10.4%	13.1%	3.8%	3.9%	4.1%	4.2%	4.6%	4.9%	3.8%	4.1%	4.2%	3.7%	3.7%	3.8%
Sharpe ratio	0.433	0.467	0.493	0.765	0.618	0.518	0.615	0.643	0.641	0.764	0.693	0.671	0.714	0.663	0.654	0.659	0.668	0.669
Sortino ratio	0.39	0.40	0.46	0.54	0.45	0.42	0.46	0.48	0.48	0.56	0.49	0.47	0.53	0.48	0.48	0.50	0.51	0.51
Max draw-down	50.4%	52.8%	37.9%	41.4%	61.7%	76.3%	43.1%	39.6%	38.6%	38.3%	49.4%	52.5%	39.8%	47.8%	49.3%	38.8%	35.3%	35.0%
VaR at 95%	7.0%	6.4%	4.5%	11.3%	15.3%	19.2%	5.1%	5.4%	5.4%	6.2%	6.8%	7.2%	5.2%	5.5%	5.5%	4.7%	4.8%	4.8%
CVaR at 95%	10.1%	10.6%	7.2%	15.1%	21.4%	27.1%	8.7%	8.6%	8.8%	9.0%	10.4%	11.1%	8.6%	9.5%	9.9%	8.3%	8.3%	8.4%
Turnover	–	3.2%	111.4%	456.8%	643.1%	831.1%	45.7%	46.7%	51.0%	195.4%	219.8%	238.2%	122.5%	126.1%	130.1%	34.8%	29.2%	28.1%
Tracking error	–	1.4%	4.1%	7.9%	10.7%	13.4%	3.4%	3.3%	3.3%	4.8%	5.3%	5.6%	3.7%	3.9%	4.0%	2.8%	2.5%	2.3%

Table 3 Overall performance of portfolios with 24-month estimation periods during 1980–2014

	Index	EqW	GMV	MV15	MV20	MV25	RB15	RB20	RB25	RE15	RE20	RE25	REf15	REf20	REf25	RE15d	RE20d	RE25d
Annual return	11.8%	12.7%	10.7%	21.2%	22.7%	23.1%	14.2%	14.4%	14.7%	14.3%	14.4%	14.6%	12.6%	12.2%	11.9%	12.9%	13.1%	13.2%
Alpha	–	0.08%	0.29%	1.23%	1.57%	1.91%	0.42%	0.42%	0.42%	0.59%	0.62%	0.64%	0.32%	0.26%	0.22%	0.27%	0.25%	0.23%
Volatility	4.5%	4.7%	3.3%	7.2%	9.8%	12.3%	3.9%	4.1%	4.2%	4.2%	4.5%	4.8%	3.9%	4.1%	4.2%	3.7%	3.9%	4.0%
Sharpe ratio	0.433	0.467	0.500	0.636	0.507	0.412	0.679	0.661	0.653	0.636	0.595	0.573	0.570	0.513	0.474	0.606	0.600	0.595
Sortino ratio	0.39	0.40	0.47	0.47	0.39	0.36	0.51	0.49	0.49	0.49	0.46	0.44	0.46	0.42	0.39	0.48	0.48	0.47
Max draw-down	50.4%	52.8%	36.0%	49.2%	68.6%	82.4%	38.5%	37.3%	36.3%	44.6%	49.7%	52.0%	48.2%	51.8%	53.7%	37.7%	37.8%	39.0%
VaR at 95%	7.0%	6.4%	4.5%	10.7%	14.6%	18.9%	4.9%	5.2%	5.1%	5.8%	6.4%	6.8%	5.1%	5.6%	5.7%	4.7%	4.7%	4.7%
CVaR at 95%	10.1%	10.6%	7.2%	15.6%	21.6%	27.4%	8.2%	8.6%	8.9%	9.1%	10.1%	10.8%	8.9%	9.6%	10.1%	8.4%	8.6%	8.8%
Turnover	–	3.2%	57.6%	263.2%	383.2%	504.8%	27.0%	30.1%	33.2%	112.0%	127.7%	139.6%	66.6%	68.2%	69.9%	21.2%	19.5%	18.7%
Tracking error	–	1.4%	4.1%	7.4%	10.0%	12.5%	3.4%	3.3%	3.3%	4.8%	5.1%	5.3%	3.5%	3.4%	3.4%	2.6%	2.3%	2.2%

and the equally-weighted portfolio have lower risk than robust portfolios. In Table 2, robust portfolios show monthly volatility below 5% with half of them being less than 4%. Similar outcome is shown in Table 3. Hence, although robust portfolios have higher returns than other conservative approaches as stressed in Sect. 4.1, robust portfolios have comparable levels of volatility to the conservative portfolios during the 35-year period starting 1980. The analysis reveals that robust portfolios do not sacrifice having low risk even though robust portfolio optimization incorporates expected asset returns in contrast to strategies that invest in the overall index, the portfolio with equal weights invested in each asset, or the portfolio with minimum variance.

4.3 Risk-adjusted return

Higher portfolio return comes at a cost; an investment with high expected return usually also has higher risk. Therefore, while it is necessary to examine the risk and return of a portfolio separately, it is also extremely valuable to look at risk and return together such as expressing the amount of portfolio return received per each unit of risk taken.

Table 2 reports the Sharpe ratio of various portfolios and the results can be viewed as an aggregate of the observations reported in Sects. 4.1 and 4.2. The higher Sharpe ratio of robust portfolios compared to the three conservative benchmarks is anticipated because robust portfolios have similar levels of risk but higher return. But robust portfolios also show comparable Sharpe ratios to mean–variance portfolios because robust portfolios tend to gain relatively high returns while only taking relatively low risk. In short, robust portfolios are arguably the most efficient portfolios; robust portfolios efficiently allocate risk in order to increase portfolio return. This observation is also found in portfolios with 24-month estimation periods as shown in Table 3.

The efficiency of robust portfolios is also noticed when examining the Sortino ratio with the MAR set to zero, where only negative returns are counted towards the risk of a portfolio. The Sortino ratios for monthly returns presented in Tables 2 and 3 show that robust portfolio optimization on average is better than mean–variance or conservative approaches for obtaining a high ratio. Especially in Table 2, all robust portfolios show Sortino ratios above 0.46, which exceeds that of the other portfolios on average.

4.4 Worst-case loss

In addition to the volatility of portfolio returns, the worst-case loss of a portfolio is essential for evaluating portfolio performance. A volatile period may temporarily hurt the performance of a portfolio, but a worst-case event may cause a permanent damage to a portfolio. In fact, the aim of robust portfolio optimization is to reduce the loss during extreme situations and it is especially important when investing long-term because it is likely to experience a market crash.

The GMV portfolio has low maximum drawdown, which is expected since it targets a portfolio with the lowest variance. From Table 2, while other methods are exposed to a maximum drawdown above 50%, only GMV and robust portfolios are safe from a large drawdown. Similar observations are shown in Table 3. In addition, VaR and CVaR at 95% from Tables 2 and 3 further demonstrate the strength of robust portfolios in controlling worst-case loss; in all cases, robust portfolios have lower VaR and CVaR than mean–variance portfolios and comparable levels with the conservative benchmarks. Although robust portfolios are not necessarily better than the GMV portfolio, we will further discuss the left-tail returns in Sect. 5.

4.5 Additional results

In addition to the measures discussed so far in this section that compute either the return or the risk of a portfolio, additional measures quantify the performance of portfolios. One example is turnover because high turnover leads to high transaction costs when managing a portfolio. The three conservative benchmark portfolios have low turnover because changes in expected asset return do not influence portfolio rebalancing as much. The average monthly total turnover of portfolios are also recorded in Tables 2 and 3 and robust portfolios certainly display much lower turnover than mean–variance portfolios and sometimes even lower turnover than the GMV portfolio. These results support that robust portfolios will have a further advantage in terms of performance due to low transaction costs.

Since relative performance is essential in managing a portfolio, tracking error is another way for measuring risk. Portfolio performance relative to the overall market or a certain benchmark is a standard practice and tracking error quantifies how closely a portfolio follows market movements. Tracking errors contained in Tables 2 and 3 again illustrate the advantage of robust portfolios. Tracking error of robust portfolios are much lower than mean–variance portfolios and comparable to the GMV portfolio meaning robust portfolios deviate less from the market index than the conventional approach.

Finally, it is interesting to compare the three versions of robust portfolios that use ellipsoidal uncertainty sets. Among the three formulations, the portfolio denoted as RE (i.e., RE15, RE20, and RE25) seem to be the most aggressive ones with the highest return, and the portfolios denoted as REd (i.e., RE15d, RE20d, and RE25d) tend to be the more robust ones with smaller volatility, worst-case loss, turnover ratio, and tracking error. Since the main objective of robust portfolio optimization is increasing robustness, REd portfolios with the diagonal assumption among the ellipsoidal models are selected for further analyses in the following section.

5 Detailed performance from 1980 to 2014

The analyses on the performance of robust portfolios from 1980 to 2014 are continued by investigating shorter investment periods. Sub-period performance of portfolios is studied by dividing the 35-year investment horizon into seven 5-year periods. Afterwards, statistical significance of some of the findings is analyzed using annual returns. For the experiments here, we focus on the 12-month estimation period and the robust portfolios denoted as RB and REd since these were the most robust ones from Sect. 4.

5.1 Sub-period performance

While the results in Sect. 4 provide a good picture of overall performance from 1980 to 2014, observing sub-periods allows us to confirm if the advantage of robust portfolios is consistently observed in shorter periods. Furthermore, shorter investment periods that include bear markets give a detailed view of portfolio performance during volatile times.

Performances during each 5-year period from 1980 to 2014 are presented in Tables 4 and 5. The bold values represent cases when the corresponding robust portfolio has better performance (e.g., higher return, lower volatility, or lower worst-case loss) than at least two out of the three conservative benchmark portfolios. The performances of MV15, RB15, and RE15d in 2005–2009 are not included in Tables 4 and 5 because it is impossible to form portfolios with an annualized standard deviation of 15% at several rebalancing periods during 2008 and 2009 due to high volatility in asset returns.

In Table 4, the aggressiveness of mean–variance portfolios is consistently shown through high return and high volatility. Robust portfolios, in contrast, have performances that are no worse than the conservative benchmark portfolios; the bold values clearly demonstrate the attractiveness of robust portfolios. In particular, all robust portfolios have higher return than the GMV portfolio in all sub-periods except for 1980–1984, and all robust portfolios have lower volatility than both the index and equally-weighted portfolio in all sub-periods except for 1985–1989. In 2000–2004, which includes the collapse of the dotcom bubble, robust portfolios are clearly the best choice in terms of both return and risk. Also, even though mean–variance portfolios have a high return in 2005–2009 due to the market performance prior to 2008, results in 2008–2009 at the bottom of Table 4 show smaller losses for robust portfolios compared to mean–variance portfolios and also display the lowest volatility for robust optimization approaches.

Table 5 summarizes the performance of portfolios as in Table 4 but concentrates on the worst-case losses. In almost all sub-periods, robust portfolios dominate the performance of the index, equally-weighted, and mean–variance portfolios. Especially since 2000, robust portfolios outperform all other approaches except for GMV portfolios, and robust portfolios are even better than the GMV portfolio during 2008-2009 as shown at the bottom of Table 5.

Combining both results from Tables 4 and 5 leads to robust portfolios being the optimal choice when considering return, risk, and worst loss. While robust portfolios are sometimes not as effective as GMV portfolios at reducing left-tail events, they compromise by achieving higher returns than GMV portfolios. Thus, as long as expected asset return is considered for portfolio selection, portfolios constructed from robust portfolio optimization seem to be the best at achieving returns while also controlling the losses.

5.2 Comparison of yearly performance

As our final analysis, t-tests are performed on yearly performance values to find statistical significance on the dominance of robust portfolios. We focus on the three measures for left-tail returns investigated in Sect. 5.1 because the primary advantage of robust portfolios should be avoiding large unexpected losses.

Tables 6, 7, and 8 present comparisons for maximum drawdown, VaR at 95%, and CVaR at 95%, respectively. Each value represents the mean difference between the benchmark portfolio (specified by the row) and the robust portfolio (specified by the column). Hence, 2.31% that appears at the upper-left of Table 6 is the mean difference between the yearly maximum drawdowns of the index and RB15, and the positive value reflects that RB15 has lower maximum drawdown on average compared to the index. Statistical significance on the outperformance of robust portfolios is also shown in the table. For mean–variance and robust portfolios, comparisons are only made for portfolios with the same levels of risk appetite coefficient.

It is very clear from Tables 6, 7, and 8 that robust portfolios have lower worst-case losses than all other portfolios except the GMV portfolio, and most of the results are statistically significant at 1 or 5% level. These observations add support to the sub-period findings in Sect. 5.2.

Table 4 Sub-period returns and volatility

		Index (%)	EqW (%)	GMV (%)	MV15	MV20 (%)	MV25 (%)	RB15 (%)	RB20 (%)	RB25 (%)	RE15d (%)	RE20d (%)	RE25d (%)
1980–1984	Period return	99	109	201	302	308	292	**161**	**150**	**145**	**166**	**162**	**159**
	Annual return	14.7	15.9	24.7	32.1	32.5	31.4	**21.1**	**20.1**	**19.6**	**21.6**	**21.3**	**20.9**
	Volatility	4.6	5.0	3.0	8.0	10.9	13.8	**3.5**	**3.9**	**4.3**	**3.9**	**4.1**	**4.2**
1985–1989	Period return	138	132	90	303	352	367	98	98	92	**136**	**142**	**145**
	Annual return	18.9	18.4	13.7	32.2	35.2	36.1	14.6	14.6	13.9	**18.8**	**19.3**	**19.6**
	Volatility	5.1	5.5	4.1	10.3	13.6	16.8	5.4	5.5	5.6	5.3	5.3	5.4
1990–1994	Period return	54	54	32	474	809	1263	**57**	**59**	**62**	**58**	**61**	**63**
	Annual return	9.0	9.0	5.8	41.8	55.5	68.6	**9.5**	**9.7**	**10.1**	**9.6**	**10.0**	**10.3**
	Volatility	3.7	4.0	2.8	8.4	11.1	13.8	**3.0**	**3.1**	**3.2**	**3.2**	**3.3**	**3.4**
1995–1999	Period return	239	134	56	313	467	638	122	**142**	**154**	109	120	126
	Annual return	27.6	18.6	9.3	32.8	41.5	49.1	17.2	**19.3**	**20.5**	15.9	17.0	17.7
	Volatility	4.2	4.2	3.3	6.8	9.4	12.0	**3.6**	**3.8**	**3.9**	**3.2**	**3.3**	**3.4**
2000–2004	Period return	− 9	55	37	28	16	1	**85**	**87**	**89**	**60**	**60**	**60**
	Annual return	− 1.9	9.1	6.5	5.1	3.0	0.2	**13.1**	**13.4**	**13.5**	**9.9**	**9.8**	**9.8**
	Volatility	4.9	4.3	3.1	6.2	8.6	10.9	**3.3**	**3.5**	**3.6**	**3.1**	**3.2**	**3.2**
2005–2009	Period return	5	16	2	–	95	66	–	**6**	**8**	–	**12**	**13**
	Annual return	1.0	3.1	0.4	–	14.2	10.7	–	**1.1**	**1.6**	–	**2.2**	**2.5**
	Volatility	4.7	5.7	3.9	–	10.0	13.2	–	**3.9**	**4.0**	–	**3.4**	**3.4**
2010–2014	Period return	107	108	107	147	156	159	**129**	**148**	**158**	**125**	**125**	**126**
	Annual return	15.7	15.7	15.7	19.8	20.7	20.9	**18.1**	**19.9**	**20.9**	**17.6**	**17.6**	**17.7**
	Volatility	3.9	4.4	2.8	6.1	8.2	10.1	**3.1**	**3.2**	**3.3**	**2.8**	**2.9**	**3.0**
2008–2009	Period return	− 19	− 15	− 17	–	− 43	− 62	–	− 21	− 20	–	− 18	− 17
	Annual return	− 4.1	− 3.3	− 3.6	–	− 10.7	− 17.5	–	− 4.6	− 4.3	–	− 3.9	− 3.7
	Volatility	6.9	8.4	5.7	–	12.0	16.0	–	**5.3**	**5.3**	–	**4.7**	**4.7**

Table 5 Sub-period worst-case performance

		Index (%)	EqW (%)	GMV (%)	MV15 (%)	MV20 (%)	MV25 (%)	RB15 (%)	RB20 (%)	RB25 (%)	RE15d (%)	RE20d (%)	RE25d (%)
1980–1984	Max drawdown	16.6	18.6	8.4	23.4	38.5	50.8	**12.6**	**15.4**	**17.7**	**14.0**	**14.7**	**15.2**
	VaR at 95	5.5	5.3	2.9	8.7	13.8	18.8	**4.3**	**4.4**	**4.4**	**4.6**	**4.8**	**4.9**
	CVaR at 95	7.8	7.8	4.3	15.0	20.0	25.0	**7.2**	8.2	9.1	**7.8**	8.1	8.4
1985–1989	Max drawdown	29.9	31.5	19.9	41.4	53.2	63.1	30.4	31.5	32.3	**28.8**	**29.8**	30.5
	VaR at 95	6.7	6.7	4.2	17.8	25.0	32.0	**6.5**	**6.5**	**6.4**	**5.5**	**5.5**	**5.5**
	CVaR at 95	12.7	13.5	11.0	25.7	32.9	39.9	15.4	15.6	15.8	14.2	14.4	14.5
1990–1994	Max drawdown	17.0	21.5	13.6	20.1	25.1	31.5	**7.0**	**7.1**	**7.1**	**12.0**	**12.5**	**12.7**
	VaR at 95	5.0	6.0	4.3	12.5	16.2	19.8	5.2	5.5	5.7	**4.5**	**4.6**	**4.6**
	CVaR at 95	7.4	8.3	5.7	13.8	17.7	21.5	**6.1**	**6.1**	**6.3**	**7.1**	**7.2**	**7.3**
1995–1999	Max drawdown	17.4	21.3	24.5	16.5	24.2	32.5	**10.9**	**11.3**	**11.1**	**14.3**	**14.8**	**15.1**
	VaR at 95	4.2	4.7	5.1	8.8	12.4	15.9	5.0	4.9	4.8	**4.6**	**4.5**	**4.4**
	CVaR at 95	8.6	9.0	6.2	10.4	14.8	19.0	**5.8**	**5.9**	**5.9**	**6.5**	**6.8**	**7.0**
2000–2004	Max drawdown	45.1	22.5	19.3	37.0	51.6	64.3	**18.8**	**18.4**	**18.3**	**21.0**	**21.0**	**20.9**
	VaR at 95	9.3	6.9	4.8	12.1	15.6	19.0	**5.2**	**5.7**	**6.0**	**4.5**	**4.7**	**4.8**
	CVaR at 95	10.0	9.5	6.5	12.7	19.0	24.8	**7.1**	**7.4**	**7.9**	**6.4**	**6.5**	**6.6**
2005–2009	Max drawdown	50.4	52.8	37.9	–	61.7	76.3	–	**39.6**	**38.6**	–	**35.3**	**35.0**
	VaR at 95	8.7	10.2	5.6	–	17.0	21.2	–	**7.4**	**6.9**	–	**6.5**	**6.7**
	CVaR at 95	12.1	14.4	11.0	–	25.2	34.3	–	**10.1**	**10.4**	–	**9.1**	**9.3**
2010–2014	Max drawdown	17.7	20.6	7.2	25.1	33.6	41.4	**8.9**	**9.9**	**10.3**	**7.3**	**8.9**	**10.4**
	VaR at 95	6.1	6.7	3.2	9.4	13.8	18.0	**4.3**	**3.9**	**4.0**	**3.0**	**3.1**	**3.3**
	CVaR at 95	7.2	8.4	4.5	11.4	15.5	19.7	**4.4**	**4.6**	**4.7**	**3.8**	**4.1**	**4.6**
2008–2009	Max drawdown	45.9	49.9	36.4	–	61.7	76.3	–	**35.4**	**34.3**	–	**30.7**	**30.2**
	VaR at 95	12.2	14.2	12.7	–	25.2	34.8	–	**10.5**	**10.8**	–	**9.2**	**9.5**
	CVaR at 95	17.2	21.0	15.1	–	36.6	51.2	–	**13.7**	**14.5**	–	**11.8**	**12.3**

Table 6 Mean difference of maximum drawdown between benchmarks and robust portfolios

	RB15 (%)	RB20 (%)	RB25 (%)	RE15d (%)	RE20d (%)	RE25d (%)
Index	2.31***	2.09**	1.74**	2.55***	2.45***	2.31***
EqW	2.86***	2.64***	2.29***	3.10***	3.00***	2.86***
GMV	− 0.43	− 0.66	− 1.01	− 0.19	− 0.30	− 0.44
MV15	7.49***	–	–	7.73***	–	–
MV20	–	13.75***	–	–	14.11***	–
MV25	–	–	19.59***	–	–	20.16***

***Significant at 1%
**Significant at 5%
*Significant at 10%

Table 7 Mean difference of VaR at 95% between benchmarks and robust portfolios

	RB15 (%)	RB20 (%)	RB25 (%)	RE15d (%)	RE20d (%)	RE25d (%)
Index	0.95**	0.77*	0.52	1.08***	1.02***	0.90***
EqW	1.34***	1.17***	0.91**	1.48***	1.41***	1.30***
GMV	− 0.70	− 0.87	− 1.13	− 0.57	− 0.63	− 0.75
MV15	5.53***	–	–	5.66***	–	–
MV20	–	10.05***	–	–	10.29***	–
MV25	–	–	14.13***	–	–	14.51***

***Significant at 1%
**Significant at 5%
*Significant at 10%

Table 8 Mean difference of CVaR at 95% between benchmarks and robust portfolios

	RB15 (%)	RB20 (%)	RB25 (%)	RE15d (%)	RE20d (%)	RE25d (%)
Index	0.95**	0.77*	0.49	1.08***	1.02***	0.90**
EqW	1.37***	1.19**	0.91**	1.50***	1.44***	1.32***
GMV	− 0.70	− 0.89	− 1.16	− 0.58	− 0.64	− 0.76
MV15	5.82***	–	–	5.95***	–	–
MV20	–	10.60***	–	–	10.85***	–
MV25	–	–	14.87***	–	–	15.28***

***Significant at 1%
**Significant at 5%
*Significant at 10%

5.3 Summary of performance

Overall, in our analyses, mean–variance portfolios have the highest return as well as the highest risk. This leads to a high Sharpe ratio because mean–variance portfolios produce high long-term returns. But robust portfolios reach matching levels of Sharpe ratio with notably less risk. The efficiency of robust portfolios will become more evident when considering the low turnover ratio compared to mean–variance portfolios. Moreover, the worst-case loss of robust portfolios is small as can be seen from maximum drawdown, VaR, and CVaR. While the GMV portfolio is often better than robust portfolios at reducing risk or large losses, the GMV portfolio is also an extreme version of robust portfolios that disregards the uncertain components, which are asset returns in this case. In summary, for our sub-period analyses, robust portfolios in the U.S. equity market are arguably the most efficient portfolios and the high efficiency is gained by producing positive return while limiting its risk and worst-case loss.

6 Conclusion

In this paper, we provide a comprehensive analysis of the performance of equity portfolios created from robust portfolio optimization in the U.S. equity market. Several robust formulations are observed and the performance is compared to the classical mean–variance portfolios as well as other conventional approaches for conservative investment such as investment in the composite market index, the equally-weighted portfolio, and the GMV portfolio. The long-term performance is investigated for the 35-year period from 1980 to 2014 but sub-periods of five years and yearly performances are also analyzed. More than 10 performance measures are collected and various estimation periods are used to provide extensive results.

Robust portfolios are shown to be one of the most efficient investment strategies. Robust portfolios are superior at efficiently allocating risk than the conservative benchmarks. Also, even though robust portfolios have much lower risk than classical mean–variance portfolios, portfolios constructed from robust portfolio optimization reveal similar levels of risk-adjusted returns to mean–variance portfolios. More importantly, the robustness of robust portfolios is confirmed through small worst-case losses. In summary, the analysis in this paper verifies the strength of robust portfolio optimization that robust portfolios have low risk and low worst-case loss due to its worst-case approach and also have high efficiency because the portfolio construction is based on the risk-return tradeoff of the mean–variance framework. While the analysis examines simple robust formulations and focuses on U.S. equities, the results nonetheless provide support for the theoretical developments of robust portfolio optimization and provide valuable insight about the performance when adopting robust optimization in portfolio management.

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